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Distribution of Extreme Points of the Error Curve of Best Approximation by Incomplete Polynomials

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Let $\Pi_{n,m}=\{(x+1)^n P_m(x): P_m(x)\in \Pi_m\}$ be the set of incomplete polynomials of type (n, m) on $[-1, 1]$. We consider the distribution of the extreme points of the error curve in the best approximation of $f\in C[-1, 1]$ with $f(-1)=0$ by incomplete polynomials of type (n, m) on $[-1, 1]$ and give a Kadec-type result for this setting. In particular, we improve the rate of convergence of the original Kadec result.

1. INTRODUCTION

Let $\Pi_{n,m}=\{(x+1)^n P_m(x): P_m(x)\in \Pi_m\}$ be the set of incomplete polynomials of type (n, m). Approximation of $f\in C[-1, 1]$ with $f(-1)=0$ by incomplete polynomials has been studied by many authors and many results have been obtained (see [4, 5, 7–9]). Analogous to Chebyshev polynomials, Saff and Varga introduced the constrained Chebyshev polynomial $T_{n,m}$ of $\Pi_{n,m}$ and proved several properties of $T_{n,m}$ in [8]. Their results are given on the interval [0, 1]. To make the presentation cleaner and to be consistent with classical results, we consider the interval $[-1, 1]$ instead of [0, 1].

DEFINITION 1.1. Let $H_{n,m}$ minimize $\{\|(x+1)^{n+m}-q\| : q\in \Pi_{n,m-1}\}$. Then $T_{n,m}=H_{n,m}/\|H_{n,m}\|$ is called the constrained Chebyshev polynomial of $\Pi_{n,m}$, where $\|f\|=\max_{-1\leqslant x\leqslant 1}|f(x)|$.

The following two theorems were given in [8] on the interval [0, 1].

THEOREM 1.2. *For each $T_{n,m}$ there exist $m+1$ distinct points $\{\xi_j^{(n,m)}\}_{j=0}^m$ with*

$$2\left(\frac{n}{m+n}\right)^2-1\leqslant \xi_0^{(n,m)}<\cdots<\xi_m^{(n,m)}=1,$$

$$T_{n,m}(\xi_j^{(n,m)})=(-1)^{m-j}=(-1)^{m-j}\,\|T_{n,m}\|,$$

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and $T_{n,m}$ *is monotone in each of* $(\xi_j^{(n,m)}, \xi_{j+1}^{(n,m)}), j=0, 1, ..., m-1, (-1, \xi_0^{(n,m)})$, $(-\infty, -1)$, *and* $(1, \infty)$.

THEOREM 1.3. *If* $P \in \Pi_{n,m}$ *and* $M = \max_{0 \leqslant k \leqslant m} \{|P(\xi_k^{(n,m)})|\}$ *then*

$$|P(x)| \leqslant M\,|T_{n,m}(x)| \qquad \textit{for all} \quad x \notin (\xi_0^{(n,m)}, 1)$$

and

$$|P^{(k)}(x)| \leqslant M\,|T_{n,m}^{(k)}(x)| \qquad \textit{for all} \quad x \notin (-1, 1), k=1, 2, ..., n+m.$$

The above points $\{\xi_j^{(n,m)}\}_{j=0}^m$ *are called the extreme points of* $T_{n,m}$.

In this paper we discuss the distribution of the extreme points and zeros of $T_{n,m}$ and the distribution of the extreme points of the error curve in the best approximation by incomplete polynomials which extends the results of Kadec in [3] for approximation by ordinary polynomials with an improved rate.

2. MAIN RESULTS

Let $\Pi_{n,m}$, $T_{n,m}$, and $\xi_j^{(n,m)}$ be defined as above over the interval $[-1, 1]$ and let $\{\sigma_j^{(n,m)}\}_{j=1}^m$ denote the unique zero of $T_{n,m}$ in each $(\xi_{j-1}^{(n,m)}, \xi_j^{(n,m)})$, $j=1, ..., m$.

The first two theorems concern the distribution of $\{\xi_j^{(n,m)}\}$ and $\{\sigma_j^{(n,m)}\}$.

THEOREM 2.1. *For the* $\xi_j^{(n,m)}, j=0, ..., m$, *and* $\sigma_j^{(n,m)}, j=1, ..., m$, *defined as above,*

$$2\left(\frac{n}{n+m}\right)^2 - 1 \leqslant \xi_0^{(n,m)} < \xi_0^{(n,m-1)} < \xi_1^{(n,m)} < \cdots < \xi_{m-1}^{(n,m)} < \xi_{m-1}^{(n,m-1)}$$
$$= \xi_m^{(n,m)} = 1 \tag{2.1}$$

and

$$\sigma_1^{(n,m)} < \sigma_1^{(n,m-1)} < \sigma_2^{(n,m)} < \cdots < \sigma_{m-1}^{(n,m-1)} < \sigma_m^{(n,m)} < 1. \tag{2.2}$$

Proof. The first inequality in (2.1) is from Theorem 1.2. Now, since $T_{n,m-1} \in \Pi_{n,m-1} \subset \Pi_{n,m}$ and $|T_{n,m-1}(\xi_i^{(n,m)})| \leqslant 1$, Theorem 1.3 implies that $|T_{n,m-1}(x)| \leqslant |T_{n,m}(x)| < 1$ for $-1 < x < \xi_0^{(n,m)}$. Hence $\xi_0^{(n,m-1)} \geqslant \xi_0^{(n,m)}$. If they are equal, let $g(x) = T_{n,m}(x) + T_{n,m-1}(x)$. Then $g(\xi_k^{(n,m)})(-1)^{m-k} \geqslant 0$, $k=0, 1, ..., m$, and $g(\xi_0^{(n,m)}) = g'(\xi_0^{(n,m)}) = 0$. By counting double zeros as two zeros, one sees that $g(x)$ has at least $m+1$ zeros in $[\xi_0^{(n,m)}, 1]$ which is impossible. Thus we have $\xi_0^{(n,m)} < \xi_0^{(n,m-1)}$ as desired.

Now suppose for some k, there is more than one extreme point of $T_{n,m-1}$ in the interval $[\xi_k^{(n,m)}, \xi_{k+1}^{(n,m)}]$. Suppose $\xi_k^{(n,m)} \leqslant \xi_j^{(n,m-1)} < \xi_{j+1}^{(n,m-1)} \leqslant \xi_{k+1}^{(n,m)}$ for some j and define

$$g(x) = \operatorname{sgn}(T_{n,m}(\xi_{k+1}^{(n,m)}))\, T_{n,m}(x) + \operatorname{sgn}(T_{n,m-1}(\xi_j^{(n,m-1)}))\, T_{n,m-1}(x).$$

Then we must have that

$$g(\xi_k^{(n,m)}) \leqslant 0, \qquad g(\xi_j^{(n,m-1)}) \geqslant 0, \qquad g(\xi_{j+1}^{(n,m-1)}) \leqslant 0, \qquad g(\xi_{k+1}^{(n,m-1)}) \geqslant 0$$

and hence $g(x)$ has at least two zeros in $[\xi_k^{(n,m)}, \xi_{k+1}^{(n,m)}]$ without counting double zeros at the endpoints. Now if $g(\xi_k^{(n,m)}) = 0$ then $g(x)$ has a double zero at $\xi_k^{(n,m)}$ and similarly for $\xi_{k+1}^{(n,m)}$ if $k+1 \neq m$. Since $g(\xi_i^{(n,m)})(-1)^{m-i} \geqslant 0$, $i = 0, 1, ..., m$, $g(x)$ has at least k zeros in $[\xi_0^{(n,m)}, \xi_k^{(n,m)}]$ and $m-k-1$ zeros in $[\xi_{k+1}^{(n,m)}, 1]$, without counting double zeros at the end points. Hence $g(x)$ has at least $m+1$ zeros in $[\xi_0^{(n,m)}, 1]$ which contradicts the fact that $0 \not\equiv g(x) \in \Pi_{n,m}$. This shows that each subinterval $[\xi_k^{(n,m)}, \xi_{k+1}^{(n,m)}]$ contains at most one $\xi_j^{(n,m-1)}$. But there are precisely m extreme points of $T_{n,m-1}$ in $(\xi_0^{(n,m)}, 1]$. Since $\xi_{m-1}^{(n,m-1)} = \xi_m^{(n,m)} = 1$ and $\xi_0^{(n,m)} < \xi_0^{(n,m-1)}$, $(\xi_{k-1}^{(n,m)}, \xi_k^{(n,m)})$ containing no $\xi_j^{(n,m-1)}$ for some $0 < k < m$ would imply $[\xi_{l-1}^{(n,m)}, \xi_l^{(n,m)}]$ containing more than one $\xi_j^{(n,m-1)}$ for some $0 < l < m$. Hence (2.1) must hold. Similar arguments will give (2.2). ∎

THEOREM 2.2. *Let $T_{n,m}$, $\{\xi_k^{(n,m)}\}_{k=0}^m$, and $\{\sigma_k^{(n,m)}\}_{k=1}^m$ be defined as above. Then*

$$\frac{(1-\xi_0^{(n,m)})}{2}\left(\cos\left(\frac{m-k}{m}\pi\right)+1\right) + \xi_0^{(n,m)} \leqslant \xi_k^{(n,m)} \leqslant \cos\left(\frac{m-k}{m+n}\pi\right) \tag{2.3}$$

and

$$\frac{(1-\xi_0^{(n,m)})}{2}\left(\cos\left(\frac{m-k+1/2}{m}\pi\right)+1\right) + \xi_0^{(n,m)} < \sigma_k^{(n,m)} < \cos\left(\frac{m-k+1/2}{m+n}\pi\right), \tag{2.4}$$

where strict inequalities hold in (2.3) *except for $k=0$, m on the left side and $k=m$ on the right side.*

Proof. We omit the superscripts in the following proofs for simplicity. Also, we only show the left side inequality of (2.3) as the proofs of the other inequalities are similar.

Define $H(x) = ((x+1)/2)^n\, T_m(2(x-\xi_0)/(1-\xi_0)-1)$, where $T_m(x) = \cos(m \cos^{-1}(x))$, the Chebyshev polynomial of degree m. Set $x_k = (1/2)(1-\xi_0)(\cos((m-k)\,\pi/m)+1)+\xi_0$, $k = 0, 1, ..., m$. Then $\{x_k\}$ consists of the extreme points of $T_m(2(x-\xi_0)/(1-\xi_0)-1)$. We first claim $x_k \leqslant \xi_k$ for all k. Suppose this does not hold for some k, i.e., $\xi_k < x_k$. Then set $G(x) = (4/(\xi_k + x_k + 2))^n H(x) - T_{n,m}(x) \in \Pi_{n,m}$. Now, we have

$$\left|\left(\frac{4}{x_k+\xi_k+2}\right)^n H(x)\right| < 1, \qquad \text{for} \quad x \in [\xi_0, \xi_k]$$

and

$$\left|\left(\frac{4}{x_k+\xi_k+2}\right)^n H(x_j)\right| > 1, \qquad \text{for} \quad j \geqslant k$$

implying that $G(x)$ has at least k zeros in $[\xi_0, \xi_k]$ and $m-k$ zeros in $[x_k, 1]$. Since $\operatorname{sgn} G(x_k) = -\operatorname{sgn} G(\xi_k)$, $G(x)$ has one more zero in (ξ_k, x_k). This shows $G(x)$ must have at least $m+1$ zeros in $[\xi_0, 1]$ which contradicts the fact that $G(x) \in \Pi_{n,m}$ does not vanish identically. Thus $\xi_k \geqslant x_k$ for all k.

Now, if $\xi_k = x_k$ for some $k \neq 0$, m, then $G(x)$ has $k-1$ zeros in (ξ_0, ξ_{k-1}), $m-k-1$ zeros in $(x_{k+1}, 1)$ and one zero at $x_k = \xi_k$. Since $|(2/(x_k+1))^n H(x)| < 1$ for $x < x_k = \xi_k$, $|(2/(x_k+1))^n H(x_k)| = 1$, and $G'(x_k) = (2/(x_k+1))^n H'(x_k) \neq 0$, there exists $\varepsilon > 0$ such that for $x \in (x_k, x_k+\varepsilon)$, $|(2/(x_k+1))^n H(x)| > 1$ and $\operatorname{sgn} H(x) = \operatorname{sgn} H(x_k)$. Choose any $x^* \in (x_k, x_k+\varepsilon)$ then we have that $\operatorname{sgn} G(x^*) = \operatorname{sgn} H(x_k) = -\operatorname{sgn} H(x_{k+1}) = -\operatorname{sgn} G(x_{k+1})$. That is, $G(x)$ has one more zero in (x^*, x_{k+1}). But

$$\begin{aligned}\operatorname{sgn} G(x_{k+1}) &= \operatorname{sgn} H(x_{k+1}) = (-1)^{m-k-1} = \operatorname{sgn} T_{n,m}(\xi_{k-1}) \\ &= -\operatorname{sgn} G(\xi_{k-1})\end{aligned}$$

implies $G(x)$ has an odd number of zeros in (ξ_{k-1}, x_{k+1}). Therefore $G(x)$ has at least 3 zeros in (ξ_{k-1}, x_{k+1}) and hence at least $m+1$ zeros in $(\xi_0, 1)$. This gives us our desired contradiction. Hence the inequality is proved. ∎

Now we turn to a Kadec-type result for approximation by incomplete polynomials. We first prove a lemma to deal with the technical parts of the theorem.

LEMMA 2.3. *Let $Q_m \in \Pi_{n,m}$ with $\|Q_m\| < L$ and suppose there exist $-1 \leqslant t_0 < \cdots < t_m \leqslant 1$ satisfying $\operatorname{sgn} Q_m(t_i) = (-1)^i \operatorname{sgn} Q_m(t_0)$ and $|Q_m(t_i)| = 1$ for $i = 0, 1, ..., m$. Set*

$$\Delta_m = \max_{0 \leqslant k \leqslant m} \{|\cos^{-1}(t_k) - \cos^{-1}(\xi_k^{(n,m)})|\}.$$

Then

$$\Delta_m \leqslant 3\left(\frac{\cosh^{-1} L}{m}\right)^{2/3} + o\left(\frac{\cosh^{-1} L + n}{m}\right)^{2/3} \tag{2.5}$$

for m sufficiently larger than $\cosh^{-1} L$ (*e.g.*, $m \geqslant (\cosh^{-1} L + 2)^6$).

Proof. Let $0 < l < m$ be an integer and define

$$H(x) = T_l\left(2\left(\frac{1-x}{1-t_k}\right) - 1\right) T_{n,m-l} \in \Pi_{n,m}$$

and

$$G(x) = \frac{H(x)}{1+\varepsilon} \pm Q_m(x),$$

where the + or − sign preceding $Q_m(x)$ will be chosen later. Then $|H(x)| \leqslant 1$ for $t_k \leqslant x \leqslant 1$. Noting the fact that $T_l(y) = \cosh(lt)$ with $y = \cosh(t)$ for $y \geqslant 1$, we have

$$\frac{T_l(2((1-x)/(1-t_k)) - 1)}{1+\varepsilon} > L$$

$$\text{for} \quad x \leqslant t_k - \frac{1-t_k}{2}\left[\cosh \frac{\cosh^{-1} L(1+\varepsilon)}{l} - 1\right] \triangleq t_k - A_k(\varepsilon).$$

Now we establish that $t_k \leqslant \xi_k^{(n,m-l)} + A_k(0)$, $k = 0, 1, ..., m-l$, by showing that $t_k \leqslant \xi_k^{(n,m-l)} + A_k(\varepsilon)$ for any $\varepsilon > 0$.

Suppose this is not true, that is, suppose $t_k > \xi_k^{(n,m-l)} + A_k(\varepsilon)$ for some $\varepsilon > 0$ and some k. Then $G(x)$ has at least $m-k$ zeros in $(t_k, 1]$ and k zeros in $(0, \xi_k^{(n,m-l)})$ since $|Q_m(t_j)| = 1$ with alternating signs, $|H(x)|/(1+\varepsilon) < 1$ in $(t_k, 1]$, and $|H(\xi_j^{(n,m-l)})| \geqslant L$ for $j \geqslant k$ with alternating signs. Now we can choose a proper sign in the definition of $G(x)$ to obtain one more zero in $(\xi_k^{(n,m-l)}, t_k)$. Doing this gives our desired contradiction. Thus we have proved

$$t_k \leqslant \xi_k^{(n,m-l)} + A_k(0), \qquad 0 \leqslant k \leqslant m-l;$$

that is,

$$t_k \leqslant \xi_k^{(n,m-l)} + \frac{(1-t_k)}{2}\left(\cosh \frac{\cosh^{-1} L}{l} - 1\right)$$

$$\triangleq \xi_k^{(n,m-l)} + (1-t_k)A, \qquad 0 \leqslant k \leqslant m-l. \tag{2.6}$$

Next we need to show that

$$\cos^{-1}(t_k) \geq \cos^{-1}(\xi_k^{(n,m)}) - 3\left(\frac{\cosh^{-1} L}{m}\right)^{2/3} - o\left(\frac{\cosh^{-1} L + n}{m}\right)^{2/3}. \tag{2.7}$$

It is easy to see that we need only establish (2.7) under the assumption $t_k > \xi_k^{(n,m-l)}$ and $\xi_k^{(n,m-l)} + (1-t_k)A < 1$. Let

$$t_k = \xi_k^{(n,m-l)} + (1-t_k)\,\theta A, \tag{2.8}$$

where $0 < \theta \leq 1$. Then

$$\cos^{-1}(t_k) = \cos^{-1}(\xi_k^{(n,m)}) + [\cos^{-1}(\xi_k^{(n,m-l)}) - \cos^{-1}(\xi_k^{(n,m)})]$$
$$+ [\cos^{-1}(\xi_k^{(n,m-l)} + (1-t_k)\,\theta A) - \cos^{-1}(\xi_k^{(n,m-l)})].$$

But from (2.3)

$$|\cos^{-1}(\xi_k^{(n,m-l)}) - \cos^{-1}(\xi_k^{(n,m)})| \leq \frac{mn + kl - kn}{m(m+n-l)}\pi$$

and by the Mean-Value Theorem, (2.3), (2.8), and for $A < 1$ we have

$$|\cos^{-1}(\xi_k^{(n,m-l)}) - \cos^{-1}(\xi_k^{(n,m-l)} + (1-t_k)\,\theta A)|$$
$$= \left|\frac{(1-t_k)\,\theta A}{\sqrt{1-(\xi_k^{(n,m-l)} + (1-t_k)\,\theta\mu A)^2}}\right|$$
$$= \frac{\sqrt{1-t_k}\,\theta A}{\sqrt{1+\xi_k^{(n,m-l)} + (1-t_k)\,\theta\mu A + (1+t_k)(1-\mu)\,\theta A - (1-t_k)(1-\mu)^2\,\theta^2 A^2}}$$
$$\leq \frac{A}{\sqrt{1+\xi_k^{(n,m-l)}}} \leq \frac{A}{\sin(k\pi/2(m-l))} \leq \frac{A}{(k/(m-l))}, \tag{2.9}$$

where $0 < \mu < 1$. Set $l = [\frac{1}{2}m^{5/6}(\cosh^{-1} L)^{2/3}/(k+1)^{1/2}]$ and assume $(\cosh^{-1} L)^6 \leq m$. Then for $1 \leq k \leq m-l$ we have that

$$\frac{mn + kl - kn}{m(m+n-l)}\pi \leq \frac{n}{m+n-l}\pi + \frac{k^{1/2}(\cosh^{-1} L)^{2/3}}{2m^{1/6}(m+n-l)}\pi$$
$$\leq o\left(\frac{n}{m+n}\right)^{2/3} + \frac{\pi}{2}\left(\frac{\cosh^{-1} L)}{m}\right)^{2/3} + o\left(\frac{\cosh^{-1} L}{m}\right)^{2/3}$$

and

$$\frac{m-l}{k}A \leqslant \frac{m-l}{2k}\left[\frac{1}{2}\left(\frac{\cosh^{-1} L}{l}\right)^2 + \sum_{j=2}^{\infty} \frac{1}{(2j)!}\left(\frac{\cosh^{-1} L}{l}\right)^{2j}\right]$$
$$\leqslant \left(\frac{\cosh^{-1} L}{m}\right)^{2/3} + o\left(\frac{\cosh^{-1} L}{m}\right).$$

Hence,

$$\cos^{-1}(t_k) \geqslant \cos^{-1}(\xi_k^{(n,m)}) - \frac{mn+kl-kn}{m(m+n-l)}\pi - \frac{m-l}{k}A$$
$$\geqslant \cos^{-1}(\xi_k^{(n,m)}) - 3\left(\frac{\cosh^{-1} L}{m}\right)^{2/3} - o\left(\frac{\cosh^{-1} L+n}{m}\right)^{2/3}. \tag{2.10}$$

For $k \geqslant m-l$ we have that

$$\cos^{-1}(t_k) \geqslant 0 \geqslant \cos^{-1}(\xi_k^{(n,m)}) - \frac{m-k}{m}\pi$$
$$\geqslant \cos^{-1}(\xi_k^{(n,m)}) - \frac{l}{m}\pi \geqslant \cos^{-1}(\xi_k^{(n,m)}) - \frac{(\cosh^{-1} L)^{2/3}}{2m^{1/6}(m-l)^{1/2}}\pi$$
$$= \cos^{-1}(\xi_k^{(n,m)}) - \frac{\pi}{2}\left(\frac{\cosh^{-1} L}{m}\right)^{2/3} - o\left(\frac{\cosh^{-1} L}{m}\right)^{2/3}.$$

For $k=0$, from Theorem 2.1, $\xi_0^{(n,m-l)} \geqslant 2(n/(n+m-l))^2 - 1$ and hence in view of (2.9)

$$\cos^{-1}(\xi_0^{(n,m-l)}) - \cos^{-1}(\xi_0^{(n,m-l)} + (1-t_k)\,\theta A)$$
$$\leqslant \frac{A(n+m-l)}{n} \qquad \text{for} \quad n \geqslant 0.$$

When $n=0$, $\xi_0^{(n,m)} = -1$ and $\cos^{-1}(-1) - \cos^{-1}(-1+(1-t_k)\,\theta A) \leqslant 2\sqrt{A}$ for sufficiently small A, and $\sqrt{A} = o(1/m)^{2/3}$. This makes (2.10) valid for $k=0$ (replace $(m-l)/k$ by $(n+m-l)A/n$ or $2\sqrt{A}$) and hence (2.7) is established for $0 \leqslant k \leqslant m$.

Similarly we can show

$$\cos^{-1}(t_k) \leqslant \cos^{-1}(\xi_k^{(n,m)}) + 3\left(\frac{\cosh^{-1} L}{m}\right)^{2/3} + o\left(\frac{\cosh^{-1} L+n}{m}\right)^{2/3}. \tag{2.11}$$

Combining (2.7) and (2.11), we obtain (2.5). ∎

We now state our main theorem.

THEOREM 2.4. *Let* $f \in C[-1, 1]$, $f(-1)=0$ *if* $n \neq 0$, *and* $f \notin \bigcup_{m=1}^{\infty} \Pi_{n,m}$. *Suppose* $P_m \in \Pi_{n,m}$ *is the best approximation to* f *from* $\Pi_{n,m}$. *Set* $E_m = \|f - P_m\|$ *and let* $-1 \leqslant x_0 < x_1 < \cdots < x_{m+1} \leqslant 1$ *be the Chebyshev alternation points of* $f - P_m$, *that is,*

$$P_m(x_i) - f(x_i) = \sigma(-1)^i E_m, \qquad \sigma = \pm 1.$$

Set

$$\Delta_m = \max_{0 \leqslant k \leqslant m+1} \{|\cos^{-1}(x_k) - \cos^{-1}(\xi_k^{(n,m+1)})|\}.$$

Then

$$\varliminf_{m \to \infty} \frac{\Delta_m m^{2/3}}{(\log m)^{2/3}} \leqslant 3. \tag{2.12}$$

Proof. Let $Q_{m+1} = (P_{m+1} - P_m)/(E_m - E_{m+1}) \in \Pi_{n,m+1}$. By the assumption that $f \notin \bigcup_{m=1}^{\infty} \Pi_{n,m}$, Q_{m+1} is well defined for infinite many m,

$$\|Q_{m+1}(x)\| \leqslant \frac{E^{m+1} + E_m}{E_m - E_{m+1}} \triangleq L_m$$

and

$$|Q_{m+1}(x_k)| \geqslant 1$$

whenever $E_{m+1} < E_m$.

Let t_k, $k = 0, 1, ..., 2m+3$, be the roots of the equation of $|Q_{m+1}(x)| = 1$ with $t_{2k} \leqslant x_k \leqslant t_{2k+1}$. Note that $Q_{m+1}(t_{2k}) = Q_{m+1}(t_{2k+1}) = -Q_{m+1}(t_{2k+2}) = -Q_{m+1}(t_{2k+3})$, $k = 0, ..., m$, and then by Lemma 2.3 we have that

$$\Delta_m \leqslant 3\left(\frac{\cosh L_m}{m+1}\right)^{2/3} + o\left(\frac{\cosh^{-1} L_m + n}{m+1}\right)^{2/3}.$$

We claim that $\varliminf_{m \to \infty} (\log L_m)/\log m^{\alpha} \leqslant 1$, for any $\alpha > 1$. Supposing this is not true, there exist m_0 and $\alpha > 1$ such that $L_m > m^{\alpha}$ for all $m \geqslant m_0$ and hence $E_{m+1} > [(m^{\alpha} - 1)/(m^{\alpha} + 1)] E_m$. Hence

$$\varlimsup_{m \to \infty} \frac{E_m}{E_{m_0}} \geqslant \varlimsup_{m \to \infty} \prod_{k=m_0}^{m-1} \left(\frac{k^{\alpha} - 1}{k^{\alpha} + 1}\right) > 0$$

contradicting the fact that $\lim_{m \to \infty} E_m = 0$. Thus there exists a subsequence $\{m_k\}$ such that

$$\lim_{k \to \infty} \frac{\log L_{m_k}}{\log m_k} = \lambda \leqslant 1.$$

Note that this guarantees that $(\cosh^{-1} L_{m_k} + 2)^6 \leqslant m_k$ holds for large k, which shows that the hypothesis of Lemma 2.3 is satisfied. Thus we have

$$\begin{aligned}\underline{\lim}_{m \to \infty} \frac{\Delta_m m^{2/3}}{(\log m)^{2/3}} &\leqslant \underline{\lim}_{k \to \infty} \frac{\Delta_{m_k} m_k^{2/3}}{(\log m_k)^{2/3}} \\ &\leqslant \lim_{k \to \infty} \frac{3(\cosh^{-1} L_{m_k}/(m_k+1))^{2/3}\, m_k^{2/3}}{(\log m_k)^{2/3}} = 3\lambda^{2/3} \leqslant 3. \quad \blacksquare\end{aligned}$$

COROLLARY 2.5. *Let* $f \in C[0, \pi]$ and Q_m *be the best approximation to* f *from the set of all even trigonometric polynomials of degree* $\leqslant m$. *Let*

$$t_0^{(m)} < t_1^{(m)} < \cdots < t_{m+1}^{(m)}$$

be a set of alternating extreme points of the error curve and set

$$\delta_m = \max_{0 \leqslant k \leqslant m+1} \left\{ \left| t_k^{(m)} - \frac{k\pi}{m+1} \right| \right\}.$$

Then

$$\underline{\lim}_{m \to \infty} \frac{\delta_m m^{2/3}}{(\log m)^{2/3}} \leqslant 3. \tag{2.13}$$

This corollary improves Kadec's result in [3] where he proved

$$\underline{\lim}_{m \to \infty} \delta_m m^{1/2-\varepsilon} = 0, \qquad \text{for any} \quad \varepsilon > 0.$$

If f is analytic on $[-1, 1]$ the rate in (2.12) can be slightly improved.

THEOREM 2.6. *Let the hypothesis of Theorem* 2.4 *hold and assume, in addition, that* f *is analytic on* $[-1, 1]$ *and* $f^{(k)}(-1) = 0$ *for* $k = 0, 1, \ldots, n-1$. *Then there exists a constant* c *such that*

$$\underline{\lim}_{m \to \infty} \Delta_m m^{2/3} \leqslant c. \tag{2.14}$$

Proof. Let L_m and E_m be defined as before. It can be easily shown (see [11]) that if a function f is analytic on $[-1, 1]$ and $f^{(k)}(-1) = 0$, $k = 0, 1, \ldots, n-1$ then $\overline{\lim}_{m \to \infty} \sqrt[m+n]{E_m} = q < 1$. This implies that

$$\underline{\lim}_{m \to \infty} \frac{E_{m+1}}{E_m} \leqslant q$$

because otherwise $E_{m+1} \geqslant q' E_m$ for all $m \geqslant m_0$ and some $q' > q$ implies $E_m \geqslant (q')^{m-m_0} E_{m_0}$ which contradicts $\overline{\lim}_{m\to\infty} \sqrt[m+n]{E_m} = q < q'$. Thus we have

$$\underline{\lim}_{m\to\infty} L_m \leqslant \underline{\lim}_{m\to\infty} \frac{1+E_{m+1}/E_m}{1-E_{m+1}/E_m} \leqslant \frac{2}{1-q}.$$

In view of the proof of Theorem 2.4 and the above inequality, (2.14) follows. ∎

In [10], $\tilde{\delta}_m = \max_{0\leqslant k\leqslant m+1}\{|t_k^{(m)} - t_{k+1}^{(m)}|\}$ was considered and it was proved that

$$\underline{\lim}_{m\to\infty} \tilde{\delta} \frac{m}{\log m} < \infty. \tag{2.15}$$

Similar results can be shown for best approximation by incomplete polynomials. Recently it was shown in [2] that

$$\underline{\lim}_{m\to\infty} \delta_m \frac{m}{\log m} < \infty \tag{2.16}$$

under the stronger hypothesis that f has an analytic continuation in the closed ellipse with *foci* ± 1 and sum of semiaxes $r > 1/\rho$, where $\rho \approx 0.12366$.... Note, it is an open question whether these rates are sharp or not. Evidently for some functions, for example, $f(x) = \sum_{n=1}^{\infty} (1/3)^n \cos(3^n \cos^{-1} x)$, $\delta_m = O(1/m)$. However, a counterexample given by G. G. Lorentz in [6] showed that lim inf cannot be replaced by limit even if we only consider the set of all entire functions.

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